3F8: Introduction to inference

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\[
p(X | Y) = \frac{p(Y | X)p(X)}{p(Y)}
\]
Examples of inference problems

- Climate science
- Object recognition
- Speech recognition
- Automated drug discovery
- Genomics
- Collaborative filtering
- Error correcting codes

Inputs:
- Solar
- Human
- Ocean
- Atmosphere
- Land
- Temperatures
- Sea and ice levels
- CO2

Physics:
- Physical constants
- CMB
- Supernovae

Observations:
- Astronomy
- Climate science

Predicted Tags:
- Architecture
- Travel
- No person
- River
- Building
- Outdoors
- Water
- Castle
- Tourism
- City
A problem where you have to **estimate unknown variables** from **known variables**.

Known variables are sometimes called **observed variables**.

Unknown variables are sometimes called **unobserved variables**.
A first inference problem: radioactive decay

- unstable particles emitted from a source, decay at a distance $x$
- $x$ follows an exponential distribution with characteristic length-scale $\lambda$

$$p(x|\lambda) = \frac{1}{Z(\lambda)} \exp\left(-\frac{x}{\lambda}\right)$$
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- $N$ events observed at $\{x_1, \ldots, x_N\}$. What is $\lambda$?

$$p(x|\lambda) = \frac{1}{Z(\lambda)} \exp \left( -\frac{x}{\lambda} \right)$$
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Ad hoc classes of approaches

Approach 1
- bin up into a histogram
  - where do we place the bins
- fit to density
  - what error measure do we minimise?

Approach 2
- construct an estimator e.g. the sample mean $\mu = \frac{1}{N} \sum_{n=1}^{N} x_n$
  - which estimator should we choose? mean, variance, higher moments?
- relate to parameters via expectation of estimator e.g. $\mu \approx \langle x \rangle = f(\lambda)$
  - small sample effects can be problematic e.g. if $\mu > \frac{1}{2}(50 + 5)\text{cm}$
A principled method: the probabilistic approach
A principled method: the probabilistic approach

1. "right answer" to any inference problem is a probability distribution

\[ p(u|o) \]
A principled method: the probabilistic approach

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unobserved variables \( u = \lambda \)

observed variables \( o = \{x_n\}_{n=1}^N \)
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2. plausibility computed using the sum and product rules of probability

   sum rule: \( p(y) = \int p(y,z)dz \)

   product rule: \( p(y,z) = p(z)p(y|z) = p(y)p(z|y) \)
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Cox (1946) showed:
- only way to perform consistent inferences
- generalisation of logic to uncertain situations
- see supplemental slides
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   Apply to radioactive decay example
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Apply to radioactive decay example

\[ p(\lambda|\{x_n\}_{n=1}^N) = \frac{1}{p(\{x_n\}_{n=1}^N)} \quad p(\lambda) \quad p(\{x_n\}_{n=1}^N|\lambda) \]

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Apply to radioactive decay example

\[ p(\lambda|x_n^{N}_{n=1}) = \frac{1}{p(x_n^{N}_{n=1})} \]

plausibility after observing data (posterior)
what we knew beforehand (prior)
what the data tell us (likelihood of parameters)
A principled method: the probabilistic approach

\[ p(\lambda | \{x_n\}_{n=1}^{N}) = \frac{1}{p(\{x_n\}_{n=1}^{N})} \ p(\lambda) \ p(\{x_n\}_{n=1}^{N} | \lambda) \]

plausibility after observing data (posterior) \( \propto \) what we knew before hand (prior) \( \times \) what the data tell us (likelihood of parameters)
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\[ p(\lambda | \{x_n\}_{n=1}^N) = \frac{1}{p(\{x_n\}_{n=1}^N)} \times p(\lambda) \times p(\{x_n\}_{n=1}^N | \lambda) \]

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Probability of decay events given decay constant \( p(\{x_n\}_{n=1}^N | \lambda) \)
(likelihood of the decay constant / what the data tell us)
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- decay events independent given decay const.

\[ p(\{x_n\}_{n=1}^N | \lambda) = \prod_{n=1}^{N} p(x_n | \lambda) \]
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Probability of decay events given decay constant \(p(\{x_n\}_{n=1}^N | \lambda)\) (likelihood of the decay constant / what the data tell us)

- decay events independent given decay const.
- each event follows a truncated exponential distribution

\[ p(x_n | \lambda) = \begin{cases} \frac{1}{Z(\lambda)} \exp\left(-\frac{x_n}{\lambda}\right) & \text{if } 5\text{cm} < x_n < 50\text{cm} \\ 0 & \text{otherwise} \end{cases} \]

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Z(\lambda) = \int_{5\text{cm}}^{50\text{cm}} \exp(x/\lambda) \, dx = \lambda[\exp(-5\text{cm}/\lambda) - \exp(-50\text{cm}/\lambda)]
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A principled method: the probabilistic approach

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Prior on decay constant (what we knew before seeing data) \( p(\lambda) \)
- subjective (depends on your knowledge)

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p(\lambda) = \begin{cases} 
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area must sum to one
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**Question:** do we need to retain entire dataset to compute posterior?

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**Question:** do we need to retain entire dataset to compute posterior?

**No:** only mean(x) & N

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$p(x|\lambda) = \frac{1}{Z(\lambda)} \exp \left( -\frac{x}{\lambda} \right)$
Density

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Likelihood of the parameters

\[ p(x | \lambda) = \frac{1}{Z(\lambda)} \exp \left(-\frac{x}{\lambda}\right) \]
Density $p(x|\lambda) = \frac{1}{Z(\lambda)} \exp(-x/\lambda)$
Likelihood of the parameters $p(x | \lambda) = \frac{1}{Z(\lambda)} \exp(-x / \lambda)$
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Likelihood of the parameters $p(x \mid \lambda) = \frac{1}{Z(\lambda)} \exp(-x/\lambda)$
Posterior distribution: \( p(\lambda | x_1) \propto p(\lambda) p(x_1 | \lambda) \)
Posterior distribution: $p(\lambda | x_1, x_2) \propto p(\lambda) \prod_{n=1}^{2} p(x_n | \lambda)$
Posterior distribution: 

\[ p(\lambda | x_1, x_2, x_3) \propto p(\lambda) \prod_{n=1}^{3} p(x_n | \lambda) \]
Posterior distribution:

\[ p(\lambda | x_1, x_2, x_3, x_4) \propto p(\lambda) \prod_{n=1}^{4} p(x_n | \lambda) \]
Posterior distribution: \( p(\lambda | x_1, x_2, x_3, x_4, x_5) \propto p(\lambda) \prod_{n=1}^{5} p(x_n | \lambda) \)
Summarising the posterior distribution

$\lambda_p(x)$

maximum a posteriori (MAP)

error-bars

$\lambda / \text{cm}$

$p(\lambda | x)$

$x \times 10^{-3}$
Summarising the posterior distribution

Gaussian approximation
Summarising the posterior distribution
Summary of probabilistic approach

- write down the probability of everything (joint distribution)
  \[ p(\{x_n\}_{n=1}^N, \lambda) = p(\lambda)p(\{x_n\}_{n=1}^N | \lambda) \]

- use Bayes’ rule (product rule) to form the posterior distribution
  \[ p(\lambda | \{x_n\}_{n=1}^N) = \frac{1}{p(\{x_n\}_{n=1}^N)} p(\lambda)p(\{x_n\}_{n=1}^N | \lambda) \]

- summarise the posterior e.g. via the maximum a posteriori (MAP) estimate
  \[ \lambda^{\text{MAP}} = \arg \max_\lambda p(\lambda | \{x_n\}_{n=1}^N) \]

- maximum likelihood estimate is recovered when using a wide uniform prior distribution
  \[ \lambda^{\text{ML}} = \arg \max_\lambda p(\{x_n\}_{n=1}^N | \lambda) \]
A second inference problem: Medical example

Alice has a test for a disease:

- $a = 1$ indicates Alice has the disease, $a = 0$ indicates she does not
- $b = 1$ indicates positive test result, $b = 0$ indicates it is negative

5% of people of Alice's age and background have the disease

Alice has the test and the result is positive

Compute the probability that Alice has the disease
A second inference problem: Medical example

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The test is 95% reliable:

- in 95% of cases of people who really have the disease, a positive result is returned
- in 95% of cases of people who do not have the disease, a negative result is obtained
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Alice has the test and the result is positive

- Compute the probability that Alice has the disease
A second inference problem: Medical example

\[
\begin{align*}
\text{disease} & \\
p(a) & \\
& a = 0 \quad .95 \quad a = 1 \quad .05
\end{align*}
\]
A second inference problem: Medical example

disease
$p(a)$

test
result
$p(b | a)$

\[
\begin{align*}
p(a) & : a = 0 \quad .95 \quad .05 \quad a = 1 \\
p(b | a) & : b = 0 \quad .95 \quad .05 \\
& \quad b = 1 \quad .05 \quad .95 
\end{align*}
\]
A second inference problem: Medical example

\[
p(a) \quad \begin{array}{c}
\text{disease} \\
p(b | a) \\
\text{test result}
\end{array}
\quad \begin{array}{c}
a = 0 \\
\text{.95} \\
\text{.05}
\end{array} \\
\quad \begin{array}{c}
a = 1 \\
\text{.05} \\
\text{.95}
\end{array}
\quad \begin{array}{c}
b = 0 \\
\text{.95} \\
\text{.05}
\end{array} \quad \begin{array}{c}
b = 1 \\
\text{.05} \\
\text{.95}
\end{array}
\]

\[
p(b, a)
\begin{array}{c|cc|}
\text{a} & 0 & 1 \\
\hline
0 & \text{.95 \times .95} & \text{.05 \times .05} \\
1 & \text{.95 \times .05} & \text{.05 \times .95}
\end{array}
\]
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\[
p(a) \quad \begin{array}{c}
a = 0 \\ \downarrow .95 \\ a = 1 \\ \downarrow .05 \\ \end{array}
\]

\[
p(b \mid a) \quad \begin{array}{cccc}
b = 0 & b = 1 & b = 0 & b = 1 \\ .95 & .05 & .05 & .95 \\ \end{array}
\]

\[
p(b, a) \quad \begin{array}{ccc}
a & 0 & 1 \\ 0 & .95 \times .95 & .05 \times .05 \\ 1 & .95 \times .05 & .05 \times .95 \\ \end{array}
\]
A second inference problem: Medical example

\[
p(a = 1 | b = 1) = \frac{p(b = 1 | a = 1)p(a = 1)}{p(b = 1)}
\]
A second inference problem: Medical example

\[
p(a = 1 | b = 1) = \frac{p(b = 1 | a = 1)p(a = 1)}{p(b = 1)}
\]

\[
= \frac{p(b = 1 | a = 1)p(a = 1)}{p(b = 1 | a = 1)p(a = 1) + p(b = 1 | a = 0)p(a = 0)}
\]
A second inference problem: Medical example

\[
p(a = 1|b = 1) = \frac{p(b = 1|a = 1)p(a = 1)}{p(b = 1)}
\]

\[
= \frac{p(b = 1|a = 1)p(a = 1)}{p(b = 1|a = 1)p(a = 1) + p(b = 1|a = 0)p(a = 0)}
\]

\[
= \frac{.95 \times 0.05}{.95 \times 0.05 + .05 \times .95} = \frac{1}{2}
\]
A second inference problem: Medical example

Test is 95% reliable, but probability of having the disease is only 50%
A second inference problem: Medical example

Alice has a test for a disease:

- $a = 1$ indicates Alice has the disease, $a = 0$ indicates she does not
- $b = 1$ indicates positive test result, $b = 0$ indicates it is negative

The test is 95% reliable:

- in 95% of cases of people who really have the disease, a positive result is returned
- in 95% of cases of people who do not have the disease, a negative result is obtained

5% of people of Alice’s age and background have the disease

Alice has the test and the result is positive
A second inference problem: Medical example

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5% of people of Alice’s age and background have the disease

Alice has the test and the result is positive

Treatment for the disease:

- $t = 1$ indicates Alice is treated, $t = 0$ indicates that she is not
A second inference problem: Medical example

Alice has a test for a disease:
- $a = 1$ indicates Alice has the disease, $a = 0$ indicates she does not
- $b = 1$ indicates positive test result, $b = 0$ indicates it is negative

The test is 95% reliable:
- in 95% of cases of people who really have the disease, a positive result is returned
- in 95% of cases of people who do not have the disease, a negative result is obtained

5% of people of Alice’s age and background have the disease

Alice has the test and the result is positive

Treatment for the disease:
- $t = 1$ indicates Alice is treated, $t = 0$ indicates that she is not
- Alice’s quality of life $R$ depends on whether she has the disease and whether she is treated:

$$
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix} = \begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
$$

Should Alice be treated?
A second inference problem: Medical example

Alice has a test for a disease:

- $a = 1$ indicates Alice has the disease, $a = 0$ indicates she does not
- $b = 1$ indicates positive test result, $b = 0$ indicates it is negative

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- in 95% of cases of people who really have the disease, a positive result is returned
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5% of people of Alice’s age and background have the disease

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- $t = 1$ indicates Alice is treated, $t = 0$ indicates that she is not
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\begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
$$

Should Alice be treated?
Bayesian Decision Theory

**posterior**

\[ p(a = 1|b = 1) = \frac{1}{2} \]
\[ p(a = 0|b = 1) = \frac{1}{2} \]

**reward**

\[
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix}
= \begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
\]
Bayesian Decision Theory

\[ p(a = 1 | b = 1) = \frac{1}{2} \]
\[ p(a = 0 | b = 1) = \frac{1}{2} \]

\[
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix} = \begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
\]

conditional reward

\[ R(t) = \sum_{a} R(a, t) p(a | b = 1) \]
Bayesian Decision Theory

\[ p(a = 1 | b = 1) = \frac{1}{2} \]
\[ p(a = 0 | b = 1) = \frac{1}{2} \]

\[
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix} =
\begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
\]

conditional reward

\[
R(t) = \sum_a R(a, t) p(a | b = 1)
\]

can separate inference and decision making

reward for action in that world

posterior probability of world

sum over possible worlds
Bayesian Decision Theory

**posterior**

\[ p(a = 1 | b = 1) = \frac{1}{2} \]

\[ p(a = 0 | b = 1) = \frac{1}{2} \]

**reward**

\[
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 3 & 5 \end{bmatrix}
\]

**conditional reward**

\[
R(t) = \sum_a R(a, t) p(a | b = 1)
\]

**conditional reward for not treating**

\[
R(t = 0) = R(a = 0, t = 0) p(a = 0 | b = 1) + R(a = 1, t = 0) p(a = 1 | b = 1) = 6 \frac{1}{2}
\]

It can separate inference and decision making.
Bayesian Decision Theory

\[ p(a = 1 | b = 1) = \frac{1}{2} \]
\[ p(a = 0 | b = 1) = \frac{1}{2} \]

\[
\begin{bmatrix}
R(a = 0, t = 0) & R(a = 0, t = 1) \\
R(a = 1, t = 0) & R(a = 1, t = 1)
\end{bmatrix}
= \begin{bmatrix}
10 & 7 \\
3 & 5
\end{bmatrix}
\]

\[
R(t) = \sum_a R(a, t) p(a | b = 1)
\]

conditional reward for not treating
\[
R(t = 0) = R(a = 0, t = 0)p(a = 0 | b = 1) + R(a = 1, t = 0)p(a = 1 | b = 1) = \frac{1}{2}
\]

conditional reward for treating
\[
R(t = 1) = R(a = 0, t = 1)p(a = 0 | b = 1) + R(a = 1, t = 1)p(a = 1 | b = 1) = 6
\]
Dutch Book Theorem

Assume you are willing to accept bets with odds proportional to the strength of your beliefs

\[ b(x) = 0.9 \] implies that you will accept a bet:

- \( x \) is true: win \( \geq 1 \)
- \( x \) is false: lose $9

If your beliefs do not satisfy the rules of probability theory (sum and product rules) there exists a set of simultaneous bets (called a "Dutch Book") which you are willing to accept, and for which you are **guaranteed to lose money no matter what the outcome**.

The only way to guard against Dutch Books is to ensure that your beliefs are coherent: i.e. satisfy the rules of probability.
Flavours of machine learning

Machine experiences a series of sensory inputs: $x_1, x_2, x_3, x_4, \ldots$

**Supervised learning**: machine also given desired outputs $y_1, y_2, \ldots$ and its goal is to learn to produce the correct output given a new input.

**Unsupervised learning**: goal of the machine is to build a model of $x$ that can be used for reasoning, decision making, predicting things, communicating etc.

**Reinforcement learning**: machine can also produce actions $a_1, a_2, \ldots$ which affect the state of the world, and receives rewards (or punishments) $r_1, r_2, \ldots$. Its goal is to learn to act in a way that maximises rewards in the long term.
Outline of the course: Regression

- Image restoration
- Opinion polls
- Pose estimation
- Power demand prediction
- Weather forecasting

...
Outline of the course: Classification

Object recognition

Speech recognition

Face recognition

The quick brown fox jumped over the lazy ...

Identity

Predicted Tags:
- architecture
- travel
- no person
- river
- building
- outdoors
- water
- castle
- tourism
- city

Spam filtering
Medical diagnosis
Drug discovery
Credit scoring
Click stream analysis
...
Outline of the course: Dimensionality Reduction

- Modelling data on/near manifolds
- Visualisation
- Understanding structure in high-dimensional data
- Preprocessing/feature learning:
  - Reducing computational complexity
  - Improving statistical efficiency

Cunningham and Yu, Nature Neuro, 2014

Modelling data on/near manifolds
- E.g. objects + transformations
  - = Non-linear manifolds

Preprocessing/feature learning:
- Reducing computational complexity
- Improving statistical efficiency

[Image of visualisation]

[Website link: http://cs.stanford.edu/people/karpathy/cnnembed/]

[Diagram of visualisation]
Outline of the course: Clustering

- network community detection
- image segmentation
- vector quantisation
- genetic clustering
- anomaly detection
- crime analysis

Campbell et al Social Network Analysis

vector quantisation
genetic clustering
anomaly detection
crime analysis
Outline of the course: Sequence modelling

I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted. A. Turing
Course books

Machine Learning: a Probabilistic Perspective
Kevin Patrick Murphy

Pattern Recognition and Machine Learning
Christopher Bishop

Bayesian Reasoning and Machine Learning
David Barber

Information Theory, Inference, and Learning Algorithms
David JC MacKay

and some examples from: An Introduction to Statistical Learning
Gareth James, Robert Tibshirani, and Trevor Hastie
Supervision plan

- **two sets of three examples classes** with Miguel and myself, 1hr
  - initial **triage** to catch common problems, go through hardest problems on each of the 3 examples sheets, provide hints
  - debug common questions arising from lectures
  - supervisors encouraged to attend too
  - mechanism for us to gauge level / focus of lectures

- **three small supervisions, plus a revision supervision** pairs / threes with course supervisors, 1hr
  - deal with **specific problems** you have
  - three supervisions on the three examples sheets, one for revision
Supplementary slide on Cox’s axioms (non-examinable)

**goal of inference:** plausibility of each parameter setting given data
or "degree of belief"

1. plausibility of each parameter setting can be represented by real numbers

2. take into account all evidence
   - can't leave out some data

3. consistency: if you can reason in more than one way, then each must lead to the same answer

4. equivalent states of knowledge imply same plausibility assignment

**Conclusion:** degrees of belief follow the rules of probability

- product rule: \( p(\lambda, x) = p(\lambda|x)p(x) = p(x|\lambda)p(\lambda) \)
- sum rule: \( p(x) = \sum_{\lambda} p(\lambda, x) \)